COMPACTNESS AND NUCLEARITY OF PSEUDO-DIFFERENTIAL OPERATORS ON $L^p(\mathbb{S}^1)$ AND $L^p(\mathbb{Z}),\ p \geq 1$

MOHAMMAD BAGHER GHAEMI

ABSTRACT. Let $\mathbb{Z}$ be the set of all integers and let $\mathbb{S}^1$ be the unit circle centered at the origin.

Suppose σ be a measurable function on $\mathbb{Z} \times \mathbb{S}^1$. Then for every sequence in $L^p(\mathbb{Z}),\ 1 \leq p < \infty$, we define the sequence $T_\sigma a$ by

$$(T_\sigma a)(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} \sigma(n, \theta)(\mathcal{F}_{\mathbb{Z}} a)(\theta) d\theta, \quad n \in \mathbb{Z},$$

where $\mathcal{F}_{\mathbb{Z}} a$ is the Fourier transform of a given by

$$(\mathcal{F}_{\mathbb{Z}} a)(\theta) = \sum_{n=-\infty}^{\infty} a(n) e^{in\theta}, \quad \theta \in [-\pi, \pi].$$

Assume τ be a measurable function on $\mathbb{S}^1 \times \mathbb{Z}$. Then for all f in $L^p(\mathbb{S}^1),\ 1 \leq p < \infty$, we define the function $T_\tau f$ on $\mathbb{S}^1$ by

$$(T_\tau f)(\theta) = \sum_{n \in \mathbb{Z}} e^{in\theta} \tau(\theta, n)(\mathcal{F}_{\mathbb{S}^1} f)(n), \quad \theta \in [-\pi, \pi],$$

where $\mathcal{F}_{\mathbb{S}^1} f$ is the Fourier transform of f given by

$$(\mathcal{F}_{\mathbb{S}^1} f)(n) = \widehat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} f(\theta) d\theta = \frac{1}{2\pi} < f(\theta), e^{in\theta} >, \quad n \in \mathbb{Z}.$$

Functions σ and τ are called symbols of pseudo-differential operators T_σ and T_τ on $\mathbb{Z}$ and $\mathbb{S}^1$ respectively.

In this article we show that if $\sigma(\theta, n) \in \mathbb{S}^1 \times \mathbb{Z}$ and there exist $C > 0,\ m \in \mathbb{R}$ such that for all $n \in \mathbb{Z},\ |\sigma(n, \theta)| \leq C(1 + n^2)^m$and and for all $f \in L^p(S^1)$ the series $\sum_{n=-\infty}^{\infty} \sigma(\theta, n) \hat{f}(n),\ \theta \in [-\pi, \pi]$ is absolutely convergent, then the pseudo-differential operator

$$T_\sigma : L^p(S^1) \longrightarrow L^p(S^1), \quad 1 \leq p < \infty$$

is compact and bounded. Then we prove similar results for pseudo-differential operators on $\mathbb{Z}$.

A necessary and sufficient condition for pseudo-differential operators from $L^{p_1}(\mathbb{S}^1)$ into $L^{p_2}(\mathbb{S}^1)$ and from $L^{p_1}(\mathbb{Z})$ into $L^{p_2}(\mathbb{Z})$ to be nuclear are presented for $1 \leq p_1, p_2 < \infty$. In the cases when $p_1 = p_2$, the trace formulas of pseudo-differential operators based on their symbols are given.

MOHAMMAD BAGHER GHAEMI
IRAN UNIVERSITY OF SCIENCE AND TECHNOLOGY, TEHRAN IRAN
E-mail address: mghaemi@iust.ac.ir