Line preserving flows in the Euler potentials formalism

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16th August 2016



Line preserving flows

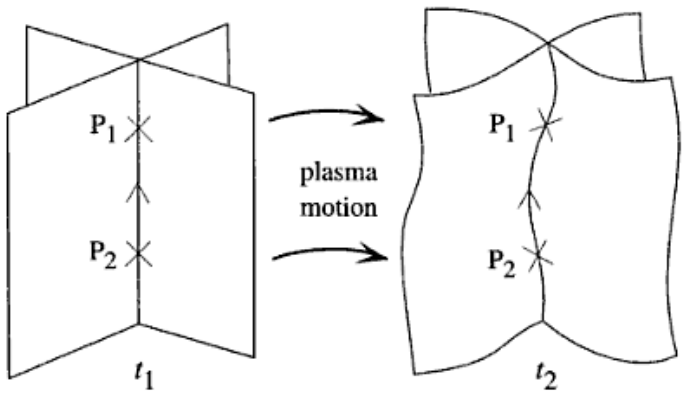


Fig. 1.7. Magnetic field-line conservation: if plasma elements P_1 and P_2 lie on a field line at time t_1, then they will lie on the same line at a later time t_2.

[E. Priest, T. Forbes, Magnetic Reconnection (2000)]

For the magnetic field $\mathbf{B}$ and the fluid velocity field $\mathbf{U}$:

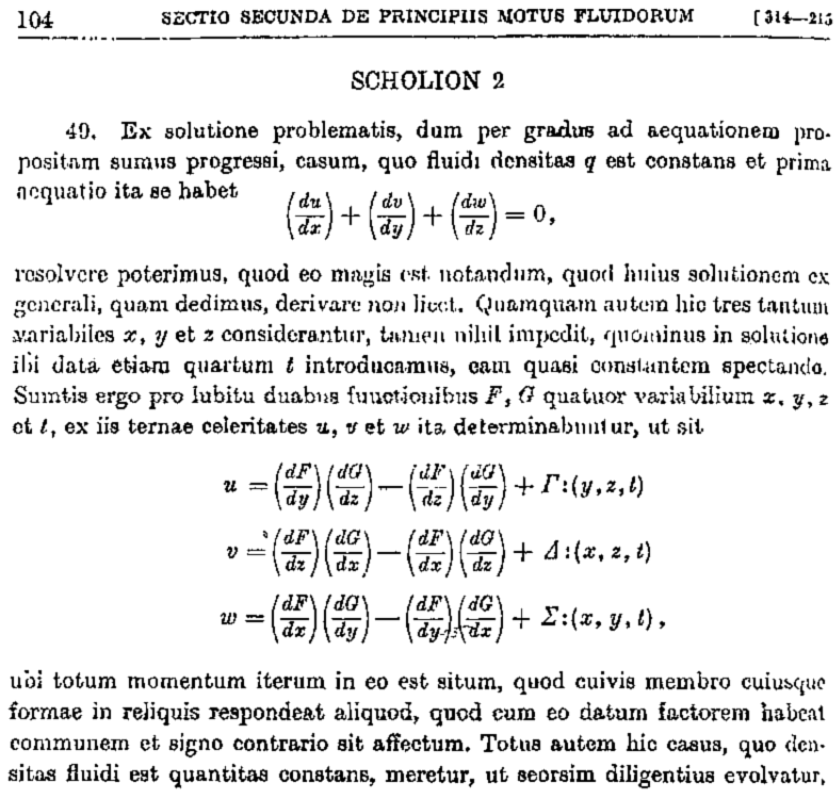
$$\mathbf{B} \times \left[\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{U} \times \mathbf{B}) \right] = 0$$

The Euler potentials
(*A.D. 1769*)

Any divergence-free vector field,
e.g. $\nabla \cdot \mathbf{B} = 0$, may be expressed as

$$\mathbf{B} = \nabla\alpha \times \nabla\beta.$$

- In this representation the field $\mathbf{B}$ is tangent to the surface families $\alpha = \text{const}$ and $\beta = \text{const}$.
- The representation is not linear; the superposition property is not valid.
- In general, the description is only local.



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SCHOLION 2

49. Ex solutione problematis, dum per gradus ad aequationem propositam sumus progressi, casum, quo fluidi densitas q est constans et prima aequatio ita se habet

$$\left(\frac{du}{dx}\right)+\left(\frac{dv}{dy}\right)+\left(\frac{dw}{dz}\right)=0,$$

resolvere poterimus, quod eo magis est notandum, quod huius solutionem ex generali, quam dedimus, derivare non licet. Quamquam autem hic tres tantum variabiles x, y et z considerantur, tamen nihil impedit, quominus in solutione ibi data etiam quartum t introducamus, eam quasi constantem spectando. Sumtis ergo pro lubitu duabus functionibus F, G quatuor variabilium x, y, z et t, ex iis ternae celeritates u, v et w ita determinabuntur, ut sit

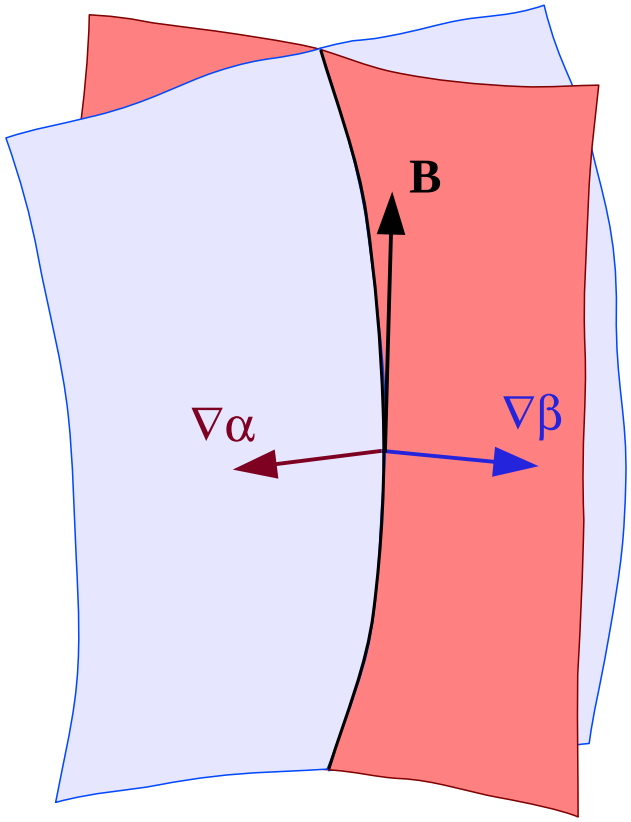
$$u=\left(\frac{dF}{dy}\right)\left(\frac{dG}{dz}\right)-\left(\frac{dF}{dz}\right)\left(\frac{dG}{dy}\right)+\Gamma:(y,z,t)$$

$$v=\left(\frac{dF}{dz}\right)\left(\frac{dG}{dx}\right)-\left(\frac{dF}{dx}\right)\left(\frac{dG}{dz}\right)+\Delta:(x,z,t)$$

$$w=\left(\frac{dF}{dx}\right)\left(\frac{dG}{dy}\right)-\left(\frac{dF}{dy}\right)\left(\frac{dG}{dx}\right)+\Sigma:(x,y,t),$$

ubi totum momentum iterum in eo est situm, quod cuivis membro cuiusque formae in reliquis respondeat aliquod, quod cum eo datum factorem habeat communem et signo contrario sit affectum. Totus autem hic casus, quo densitas fluidi est quantitas constans, meretur, ut seorsim diligentius evolvatur,

"Geometry" of the potentials



$$\mathbf{B} = \nabla\alpha \times \nabla\beta$$

The **red** surface corresponds to $\alpha = \text{const.}$

The **blue** surface corresponds to $\beta = \text{const.}$

Line preserving flows equation

$$\mathbf{U} \times \mathbf{B} = \frac{\partial \alpha}{\partial t} \nabla \beta - \frac{\partial \beta}{\partial t} \nabla \alpha - \zeta \, \alpha \nabla \beta + \nabla \Lambda \, ,$$

where ζ and Λ are arbitrary functions of α and β.

Any line preserving flow having velocity field $\mathbf{U}$ within external field $\mathbf{B}$ satisfies this equation.

Dynamics of the Euler potentials

$$\begin{aligned} \frac{d\alpha}{dt} &= -\frac{\partial \Lambda}{\partial \beta} + \zeta\alpha \\ \frac{d\beta}{dt} &= \frac{\partial \Lambda}{\partial \alpha} \end{aligned}$$

- The system of equations is *nearly-Hamiltonian*.
- In general, closed form solutions do not exist.

Solution example: linearization

Expanding functions Λ and ζ into Taylor series around point (α_0, β_0, t_0), and leaving linear terms only gives

$$\begin{array}{rcl} \frac{d\alpha}{dt} & = & -K_\beta + \alpha C \\ \frac{d\beta}{dt} & = & K_\alpha \end{array} \quad ,$$

where K_α, K_β, and C are constants. It yields following solution

$$\begin{array}{rcl} \alpha(t) & = & D_\alpha \, e^{Ct} + \frac{K_\beta}{C} \\ \beta(t) & = & K_\alpha \, t + D_\beta \end{array} \quad ,$$

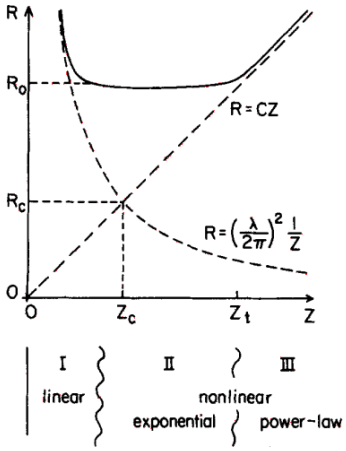
where D_α and D_β are constants. It is satisfied, for instance, by

$$\begin{array}{rl} \mathbf{B} & = [\, B_x, \, 0 \, , \, 0 \,] \\ \mathbf{U} & = [\, 0 \, , \, DC \exp{(Ct)} \, , \, 0 \,] \end{array} \quad ,$$

where B_x, C, and D are constants.

Solution example: linearization (cont.)

The solution may be applied at the early stages of Sun's coronal magnetic loop expansion.



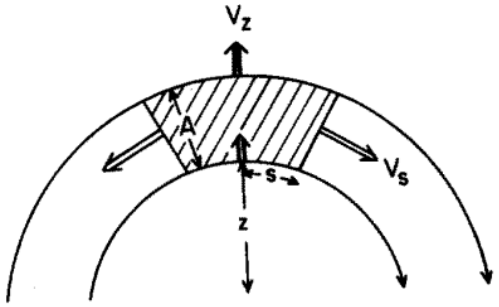
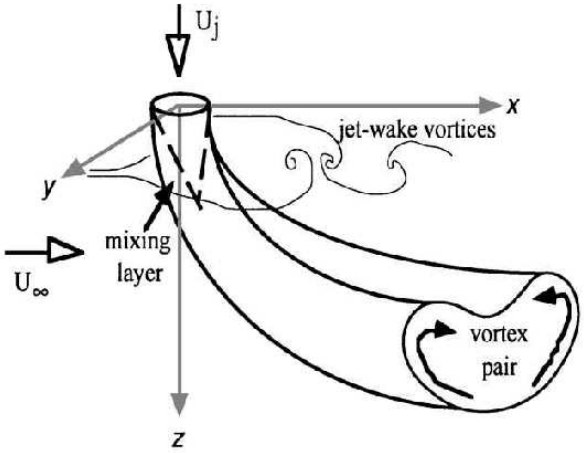


FIG. 9. An idealized model for the rising magnetic loop with downflow along it as a result of gravity.

*[K. Shibata, T. Tajima, R. Matsumoto, Phys. Fluids B **2**, 9, 1989-2000 (1990)]*

Also convenient in the jet in a crossflow problem.



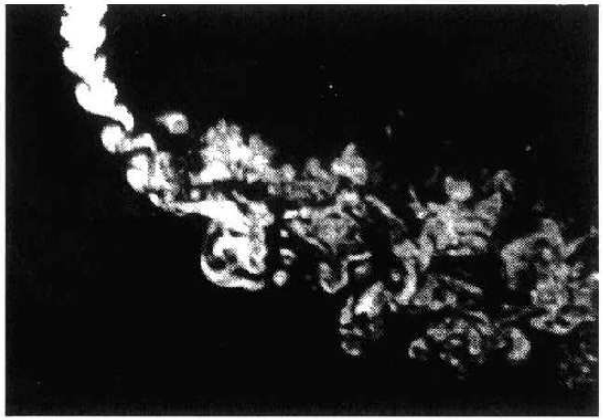


Fig. 1 Sketch and an internal, streamwise LIF picture of the round turbulent jet in a crossflow.

*[A. Eroglu, R.E. Breidenthal, AIAA J. **36**, 6, 1002-1009 (1998)]*

Solution example: finding Lagrangian

Supposing that $\zeta = \text{const}$ and that Λ is separable, *i.e.*

$$\Lambda(\alpha, \beta, t) = k(\alpha)\, l(\beta)\, r(t) \quad ,$$

and also $k(\alpha)$ and $l(\beta)$ may be expressed as series

$$k(\alpha) = \sum_{m=0}^{\infty} \frac{a\, c^m\, \alpha^{4m}\, \Gamma(\frac{3}{4})}{4^{2m}\, m!\, \Gamma(m + \frac{3}{4})}$$

$$l(\beta) = \sum_{m=0}^{\infty} \frac{b\, c^m\, \beta^{4m}\, \Gamma(\frac{3}{4})}{4^{2m}\, m!\, \Gamma(m + \frac{3}{4})} \ ,$$

where a, b i c are constants, we have that the dynamics system of equations is described by the Lagrangian

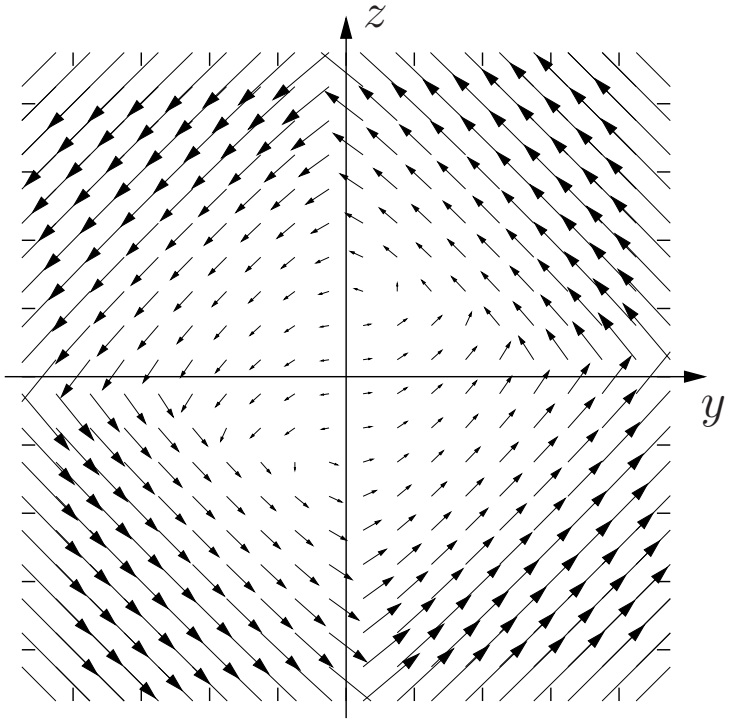
$$L = i\alpha^2_{(\frac{1}{2})} + i\beta^2_{(\frac{1}{2})} + \zeta\alpha^2 - 2\alpha\frac{\partial\Lambda}{\partial\beta} + 2\beta\frac{\partial\Lambda}{\partial\alpha} \ ,$$

i.e. the Euler-Lagrange equations reproduce the Euler potentials dynamics equations.

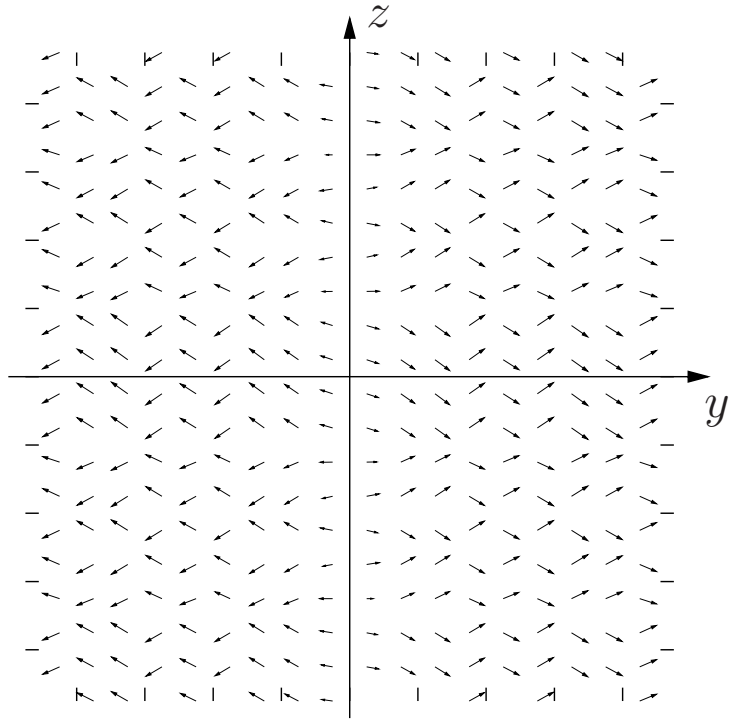
Solution example: finding Lagrangian (cont.)

Supposing the form of magnetic field $\mathbf{B} = [B_x, 0, 0]$ and using the line preserving flows equation we obtain various flow velocity fields depending on the values of constants. For example (depending on the sign of one constant only):

vortex-like flow



diffusion-like outflow



*[P. Figura, W.M. Macek, Z. Naturforsch. A Phys. Sci. **70**, 8, 643-651 (2015)]*

Perturbations

Considering equation describing deviation from the line preservation regime, *i.e.*

$$\mathbf{B} \times \left[\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{U} \times \mathbf{B}) \right] = \mathbf{W}$$

and introducing perturbations of $\mathbf{U}$ and $\mathbf{B}$ into it we obtain, among others, equations for the steady configurations when only one quantity is perturbed either $\mathbf{B}$ (while $\Delta\mathbf{U} = 0$)

$$\Delta\mathbf{B} \times [\nabla \times (\mathbf{U}_0 \times \mathbf{B}_0)] + (\mathbf{B}_0 + \Delta\mathbf{B}) \times [\nabla \times (\mathbf{U}_0 \times \Delta\mathbf{B})] = -\mathbf{W}$$

or $\mathbf{U}$ (while $\Delta\mathbf{B} = 0$)

$$\mathbf{B}_0 \times [\nabla \times (\Delta\mathbf{U} \times \mathbf{B}_0)] = -\mathbf{W}.$$

Similarly, perturbing line preserving flows equation we obtain, among others, equations for perturbed $\mathbf{B}$ (while $\Delta\mathbf{U} = 0$)

$$\mathbf{U}_0 \times \Delta\mathbf{B} = -\Delta\zeta \, \nabla\beta_0 \, \alpha_0 - \zeta_0 \, \nabla\beta_0 \, \Delta\alpha + \nabla\Delta\Lambda$$

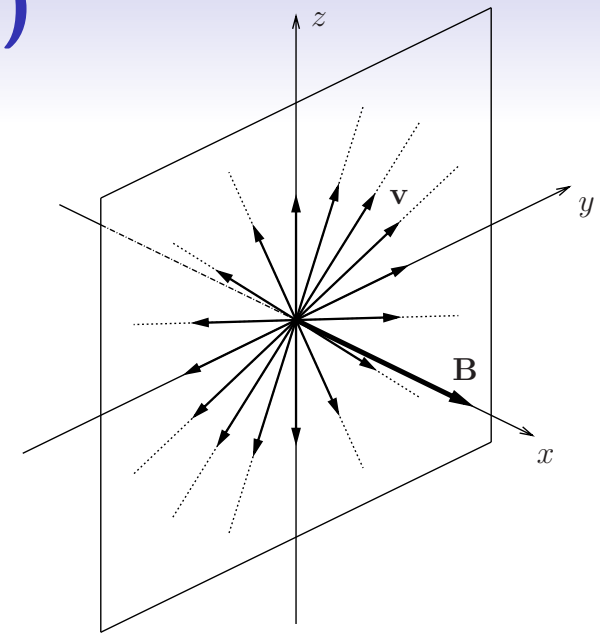
and perturbed $\mathbf{U}$ (while $\Delta\mathbf{B} = 0$)

$$\Delta\mathbf{U} \times \mathbf{B}_0 = -\Delta\zeta \, \nabla\beta_0 \, \alpha_0 - \zeta_0 \, \nabla\beta_0 \, \Delta\alpha + \nabla\Delta\Lambda.$$

Perturbations (cont.)

Source with velocity potential $\varphi(y,z) = -Q\,/\,\sqrt{y^2+z^2}$ *i.e.* $\mathbf{U}_0 = \left[\,0\,,\ \frac{Qy}{(y^2+z^2)^{3/2}}\,,\ \frac{Qz}{(y^2+z^2)^{3/2}}\,\right]$ within magnetic field $\mathbf{B}_0 = [\,ab\,,\,0\,,\,0\,]$. Equivalent with

$$\begin{cases} \zeta_0 &= \frac{-2Qab^3}{\beta^2(a^2\beta^2+b^2\alpha^2)^{1/2}} \\ \Lambda_0 &= \frac{2Qab^3\alpha}{\beta(a^2\beta^2+b^2\alpha^2)^{1/2}} \end{cases} .$$



[P. Figura, W.M. Macek, Ann. Phys. ***333****, 127-135 (2013)]*

Considering deviation from the line preservation regime and perturbing the magnetic field only we obtain

$$\lim_{\substack{x\to\infty \\ y\to\infty}} \|\mathbf{W}\| = 0.$$

Perturbing the velocity field only we have

$$\mathbf{B}_0 \times \left[B_x \frac{\partial}{\partial x}\Delta\mathbf{U}\right] = -\mathbf{W}.$$

Using the line preserving flows equation and perturbing velocity field only it appears that *e.g.* if we want to consider the system within the gravitational field, we may treat the field as a perturbation of the velocity field, *i.e.*

$$\mathbf{U} = \mathbf{U}_0 + \begin{bmatrix} 0 \\ \sqrt{2gy} \\ 0 \end{bmatrix},$$

which simplifies the considerations.

References

P. Figura, W. M. Macek, *Model of line preserving field line motions using Euler potentials*, Annals of Physics **333** (2013) *pp. 127–135*.

P. Figura, W. M. Macek, *Dynamics of Line Preserving Field Line Motions*, Zeitschrift für Naturforschung A. A Journal of Physical Sciences **70**, 8 (2015) *pp. 643–651*.

Thank you for your attention.