Ariel University

אוניברסיטת אריאל בשומרון

Faculty of Engineering

הפקולטה להנדסה



A Finite Element Approach Derived from the Simplified Variational Principle

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FlowSim
Computational Fluid Dynamics (CFD)

Outline

- Variational Principles in Physics
- Applications Computational Fluid Dynamics
- The Method of Solution
- The FLUIDEX Package
- Test Cases
- Simplified Variational Principles
- Finite Elements Approach
- Conclusions

Variational Principles in Physics

- A functional can be used to derive equations using a Variational technique
- Equations are then solved
  - analytically (if possible)
  - numerically

However

- The functional can be used **directly** for a numerical solution (simulation) by finding the extremum value of the relevant functional

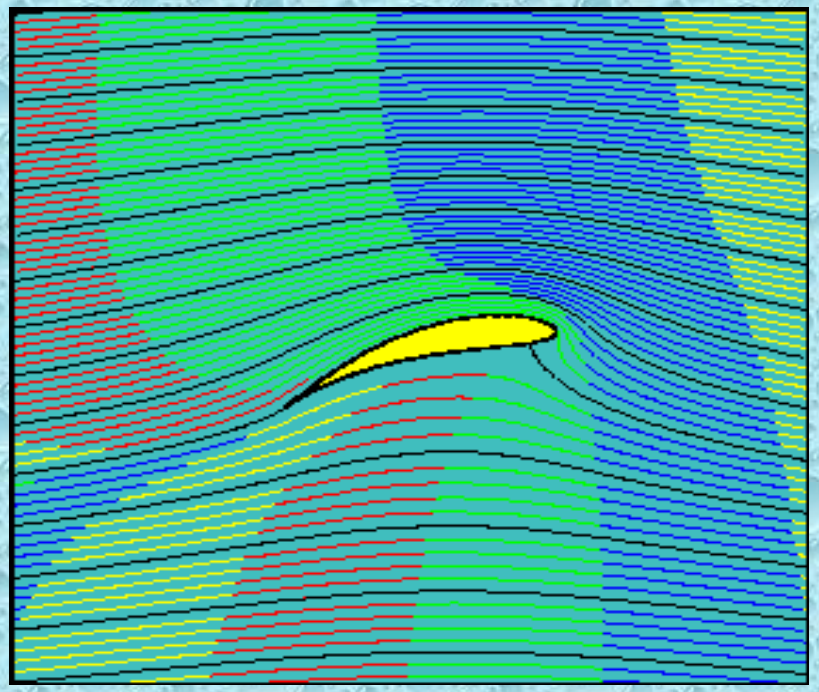
The Advantages

- Avoiding the inversion of a N^2 Matrix
  - N = number of variables × number of grid points
- Saving computer resources
  - such as CPU and memory.
- Simplicity in defining the problem over complex geometry
  - The variational principle contains both the equations and the boundary conditions

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Computational Fluid Dynamics CFD

- External flow analysis
- Flow field – Basic Variables
  - Velocity vector **u**
  - Density ρ
  - Pressure *P*
- Drag and Lift –Derived Variables
  - Forces: F_x, F_y
  - Coefficients: C_P, C_D, C_L



Equations of Motion

- Continuity

$$\frac{\partial \rho}{\partial t} + \nabla \circ (\rho \mathbf{u}) = 0$$

- Momentum: The Euler equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \circ \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla P$$

- Energy: Bernoulli's equation
  - for potential flows (derived)

$$h + \frac{v^2}{2} = C_B$$

Variational variables for the flow field

Clebsch's representation

- Velocity **u**
  - scalars: α,β,Φ

$$\mathbf{u} = \nabla\Phi + \alpha\nabla\beta$$

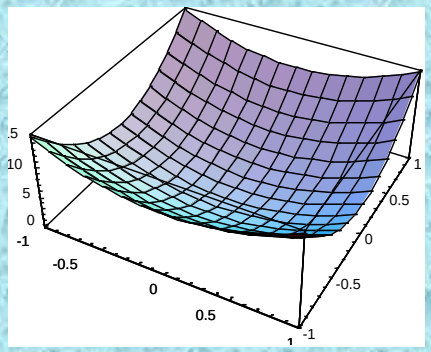
- For Irotational stationary flow:
  - $\alpha,\beta = 0$
  - potential: $\Phi = \hat{\Phi} - C_B t$

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The Variational Formulation

Fluid Dynamics

- The action, *A*, and the Lagrangian, *L*
  - Seliger and Whitham (1968)



$$\mathbf{u} = \nabla\Phi + \alpha\nabla\beta$$

$$A = \int_{t_0}^{t_1} L\,dt$$

$$L = -\int_V \rho\left[\left(\alpha\frac{\partial\beta}{\partial t} + \frac{\partial\Phi}{\partial t}\right) + \frac{1}{2}\left(\nabla\Phi + \alpha\nabla\beta\right)^2 + e(\rho) + \Omega\right]dV$$

- The solution

$$A \rightarrow extremum$$

Previous Art

Akin Ecer and collaborators have used Clebsch variables to study numerically stationary fluid problems in particular transonic flows.

Ecer has shown that such variables are useful for cases that the flow is "almost" potential.

However, Ecer et al. **did not** use the variational principle directly but rather solved the equations **derived from** the variational principle using various relaxation schemes.

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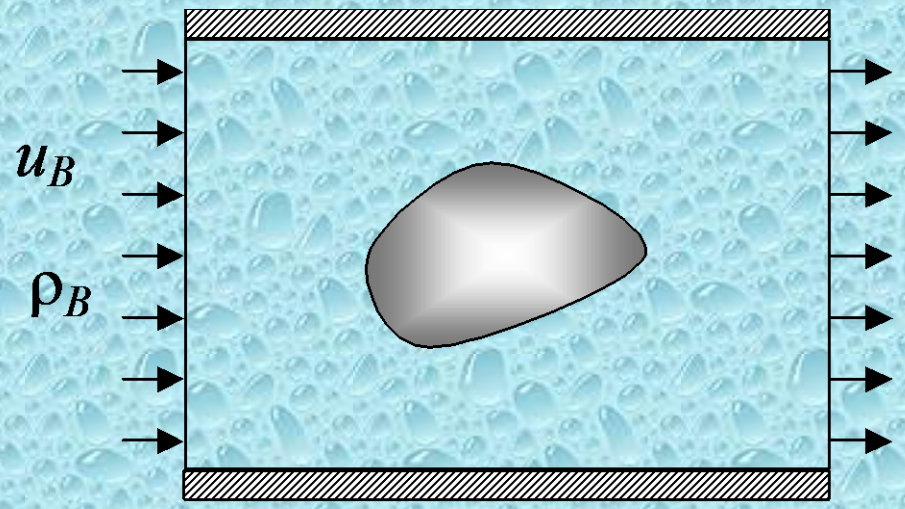
The Objective

- To develop a new method to solve the Euler flow equations numerically using variational principle technique

- "Method and System for Numerical Simulation of Fluid Flow"
  - US patent 6,516,292 (2003)

Test Case

- Steady
- Inviscid
- Irotational
- Compressible
- Baratropic
  - (or Isentropic)



The Action

- The action,

$$L = \int_V \rho \left[\frac{1}{2} \left| \nabla \hat{\Phi} \right|^2 + e - C_B \right] dV - \oint_S \hat{\Phi} \rho_B \left(\mathbf{u}_P \circ d\mathbf{s} \right)$$

$e(\rho)$- the specific internal energy

C_B-is the Bernoulli's constant

(constant for the entire field)

$\alpha = \beta = 0$ $\quad \Phi = \hat{\Phi} - C_B t$

The first variation

$$\delta L = \int_V \left[\rho\left(\nabla\hat{\Phi}\right)\circ\nabla\left(\delta\hat{\Phi}\right) + \left(\frac{1}{2}\left|\nabla\hat{\Phi}\right|^2\right)\delta\rho + \frac{\partial(e\rho)}{\partial\rho}\delta\rho - C_B\delta\rho \right] dV$$

$$-\oint_S \delta\hat{\Phi}\rho_B\left(\mathbf{u}_P\circ d\mathbf{s}\right)$$

The flow equation: $\delta L=0$

- $\delta\Phi$ internal:

  continuity equation

$$\nabla\left(\rho\nabla\hat{\Phi}\right)=0$$

- $\delta\Phi$ boundary:

  Boundary condition

$$\oint_S \left(\rho\nabla\hat{\Phi}-\rho_B\mathbf{u}_P\right)\circ d\mathbf{s}=0$$

- $\delta\rho$:

  Bernoulli's eq.

$$h+\frac{1}{2}\left|\nabla\hat{\Phi}\right|^2=C_B$$

The second variation

$$\delta^2 L = \int_V \left[\begin{array}{l} (\delta\rho)\left(\nabla\hat{\Phi}\right)\circ\delta\left(\nabla\hat{\Phi}\right)+\rho\delta\left(\nabla\hat{\Phi}\right)\circ\delta\left(\nabla\hat{\Phi}\right) \\ +\delta\left(\frac{1}{2}\left|\nabla\hat{\Phi}\right|^2+\frac{\partial(e\rho)}{\partial\rho}\right)\delta\rho \end{array} \right] dV$$

$$\delta^2 L = \left[(\delta\vec{u})+\left(\frac{\vec{u}}{\rho}\right)(\delta\rho)\right]^2+\left(a^2-\vec{u}^2\right)\left[\frac{(\delta\rho)}{\rho}\right]^2$$

- For $u<a$: $\delta^2 L>0$
  - (a is the velocity of sound)

  Extremum is also a global **minimum**!

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The Method

- Find the solution by searching for the extremum of the Action with respect to the discrete variable fields.

- Discritizasion Method:
- Minimization Method:

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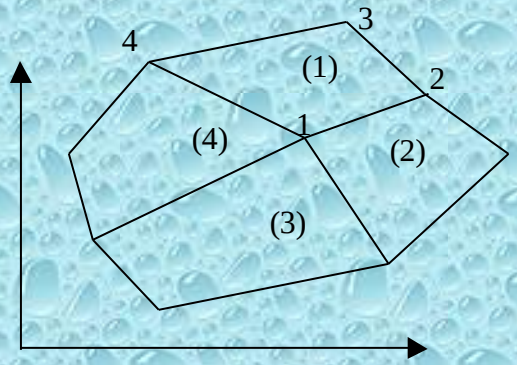
Numerical implementation of the variational principle

- The action,

$$L = -\sum_{el} \int_{V_{el}} \rho \left[\frac{1}{2} \left| \nabla \hat{\Phi} \right|^2 + e - C_B \right] dV_{el} + \frac{1}{2} \sum_{Bound.} \rho_B \hat{\Phi} \left(\mathbf{u}_p \circ \hat{\mathbf{n}}^{\alpha} \right) \left| \Delta A \right|$$

where

$$\int_{el} \left| \nabla \hat{\Phi} \right|^2 dA = \sum_{i,j} \Phi_i^e \Phi_j^e k_{ij}^e$$

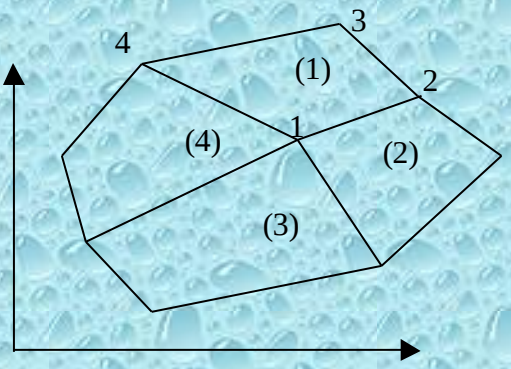


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The Method of Solution

Finite Elements method

- Nodes: potential variables
- Elements: flow variables
- Boundary terms
  - Boundary conditions



- Minimization of Functions

$$L = L\left(\Phi_i, \rho^e\right)$$

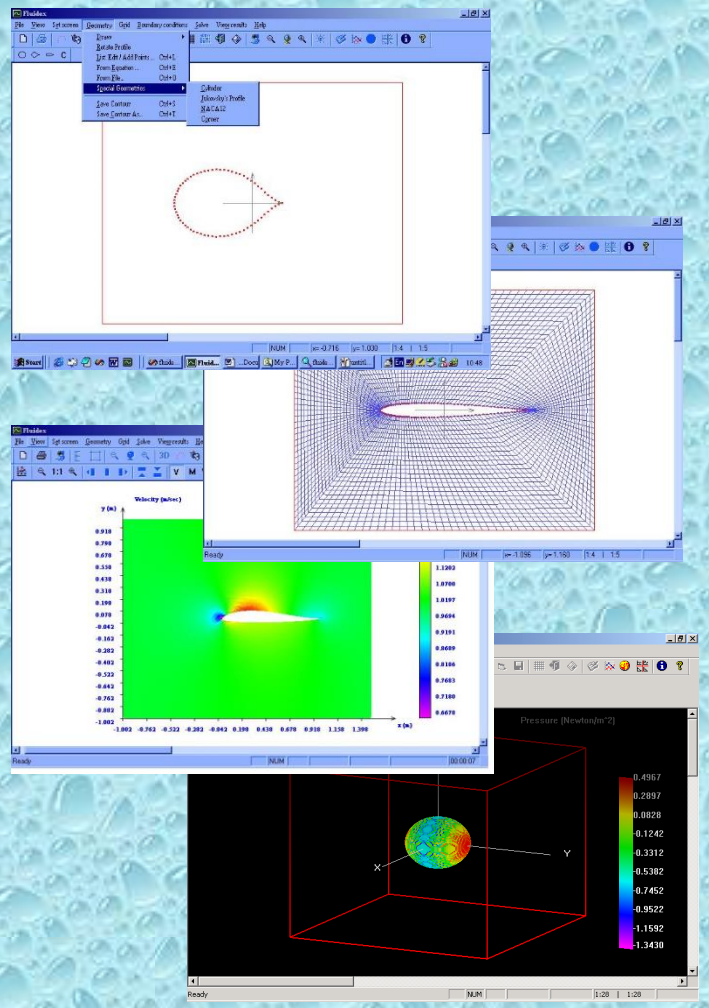
  - Conjugate gradient methods in multi-dimensions

$$\delta L_{\Phi_i} = \frac{\partial L}{\partial \Phi_i} \qquad \delta L_{\rho^e} = \frac{\partial L}{\partial \rho^e}$$

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The FLUIDEX Code

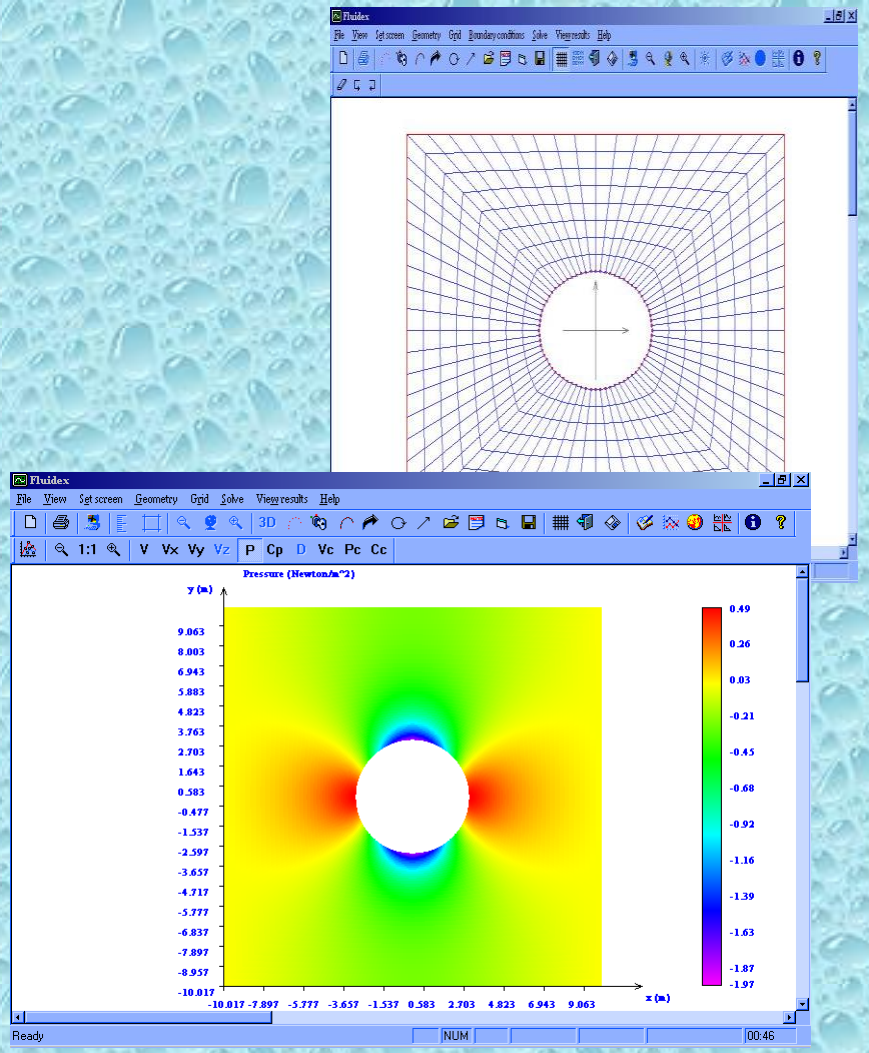
- CFD software
  - www.fluidex-cfd.com
- Platform: PC
- Complete package
  - Pre-Processing
  - Grid Generation
  - Solver
  - Post-Processing



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Test Cases

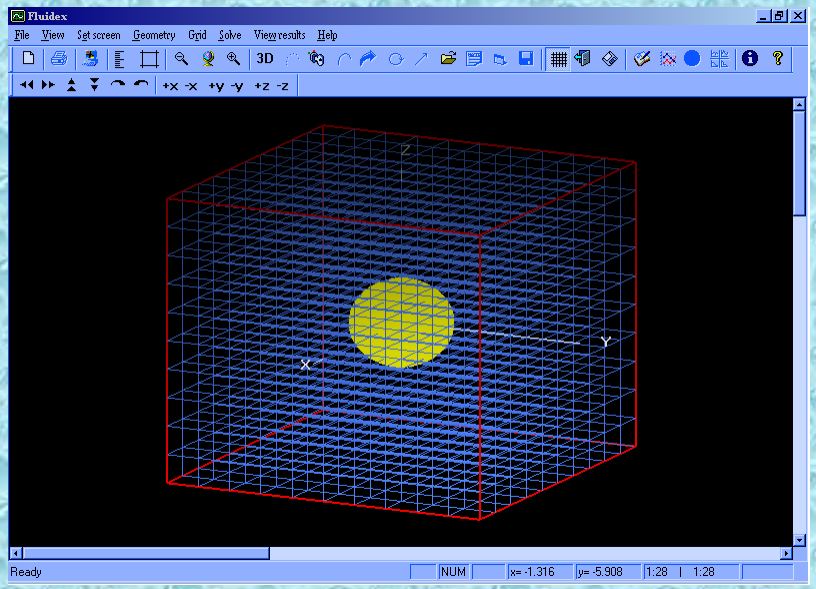
- Potential Flow
  - Cylinder and Sphere
- Compressible flow
  - Ellipse
- Airfoils
  - Joukovsky profile
  - NACA 12 profile
- Running time analysis

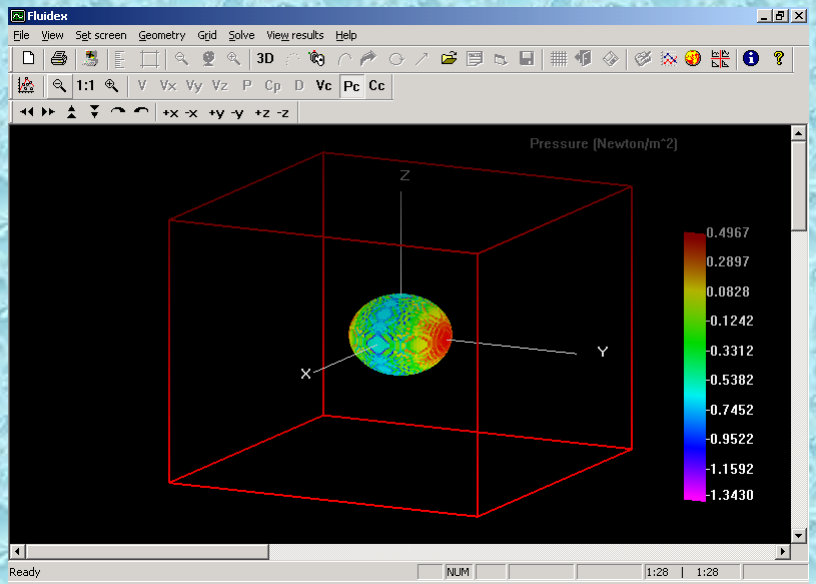


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3D Potential Flow

- Body representation
  - STL format
- The Grid
  - Modified Cartesian grid
- Pressure map
  - on the surface



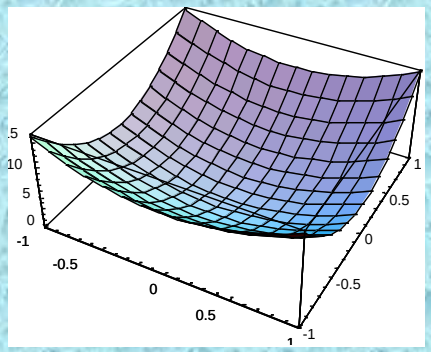


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$$A = \int_{t_0}^{t_1} L\,dt$$

$$L = -\int_V \rho\left[\left(\alpha\frac{\partial\beta}{\partial t} + \frac{\partial\Phi}{\partial t}\right) + \frac{1}{2}\left(\nabla\Phi + \alpha\nabla\beta\right)^2 + e(\rho) + \Omega\right]dV$$

- The solution

$$A \rightarrow extremum$$

Number of variational functions

In Seliger and Whitham formalism the number is four:

$$\alpha, \beta, \Phi, \rho$$

Three Clebsch variables and the density.

Can you do better than this? Can you have less functions?

Lynden-Bell & Katz (1981) say yes!

They have described a variational principle in terms of two functions the load λ (to be described below) and density ρ.

However, they “cheated”.

Their formalism contains an implicit definition for the velocity such that one is required to solve a **partial differential equation** in order to obtain both the velocity in terms of ρ and λ as well as its variations.

So the question still remains!

Until lately…..

Yahalom & Lynden-Bell (2014) have shown that three functions are enough. Those are the load λ, density ρ and a new function ν.

This is more than the two function suggested by Lynden-Bell & Katz (1981) .

But in Yahalom & Lynden-Bell (2014) formalism the functions are **independent** and are not related to each other by any partial differential equation.

How does the velocity looks in the new representation?

$$\vec{v} = -\frac{\frac{\partial \lambda}{\partial t}}{|\vec{\nabla}\lambda|}\hat{\lambda} + \vec{\nabla}\nu - \hat{\lambda}(\hat{\lambda}\cdot\vec{\nabla}\nu)$$

$$\hat{\lambda} = \frac{\vec{\nabla}\lambda}{|\vec{\nabla}\lambda|}$$

How does the Lagrangian density look in the new variables?

$$\mathcal{L} = \hat{\mathcal{L}} + \mathcal{L}_{boundary}$$

$$\hat{\mathcal{L}} \equiv \rho \left[\frac{1}{2} \left(\frac{\frac{\partial \lambda}{\partial t} + \vec{\nabla}\lambda \cdot \vec{\nabla}\nu}{|\vec{\nabla}\lambda|} \right)^2 - \frac{1}{2}(\vec{\nabla}\nu)^2 - \frac{\partial \nu}{\partial t} - \varepsilon(\rho) \right]$$

$$\mathcal{L}_{boundary} \equiv \frac{\partial(\nu\rho)}{\partial t} + \vec{\nabla} \cdot (\nu\rho\vec{v})$$

Notice that the specific choice of the labeling of the λ surfaces is not important in the above Lagrangian density one can replace: λ− > Λ(λ), without changing the Lagrangian functional form.

$$\mathcal{L} = \hat{\mathcal{L}} + \mathcal{L}_{boundary}$$

$$\hat{\mathcal{L}} \equiv \rho \left[\frac{1}{2} \left(\frac{\frac{\partial \lambda}{\partial t} + \vec{\nabla}\lambda \cdot \vec{\nabla}\nu}{|\vec{\nabla}\lambda|} \right)^2 - \frac{1}{2}(\vec{\nabla}\nu)^2 - \frac{\partial \nu}{\partial t} - \varepsilon(\rho) \right]$$

$$\mathcal{L}_{boundary} \equiv \frac{\partial(\nu\rho)}{\partial t} + \vec{\nabla} \cdot (\nu\rho\vec{v})$$

This means that only the shape of the λ surface is important not their labeling.

In group theoretic language this implies that the Lagrangian is invariant under an infinite symmetry group and hence should posses an infinite number of constants of motion.

The equations for the new variables

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0$$

$$\frac{d\nu}{dt} = \frac{1}{2}\vec{v}^2 - w$$

$$w = \frac{\partial(\rho\varepsilon)}{\partial \rho}$$

Specific Enthalpy

A third equation written in terms of a conserved quantity

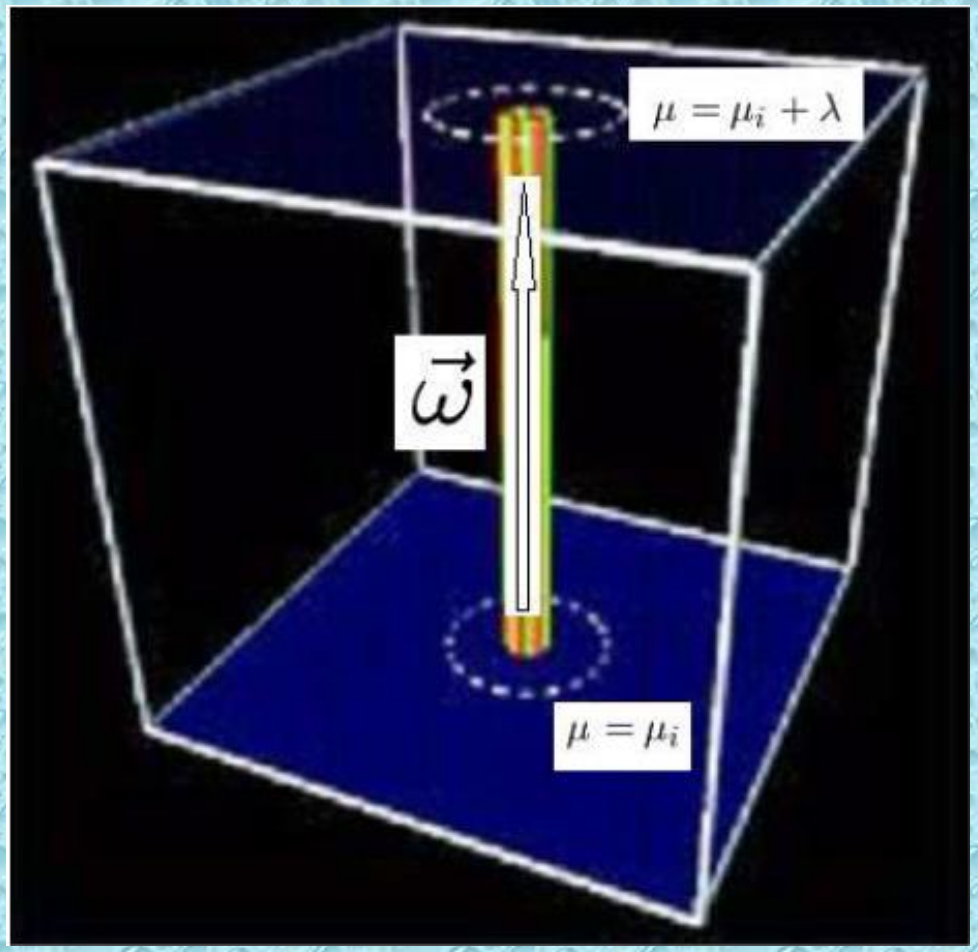
$$\theta = -\left(\frac{\frac{\partial \lambda}{\partial t} + \vec{\nabla}\lambda \cdot \vec{\nabla}\nu}{|\vec{\nabla}\lambda|^2}\right)$$

$$\rho \frac{d\theta}{dt} = 0$$

Those equations can be shown to be equivalent to the Euler equations

What is Load?

Consider the vortex tube:



Vortex Flux in the tube = Circulation around the tube

$$\Delta\Phi = \int \vec{\omega} \cdot d\vec{S} = \oint \vec{v} \cdot d\vec{r}$$

This is a conserved quantity

Another conserved quantity is the mass in the tube

$$\Delta M = \int \rho d\vec{l} \cdot d\vec{S}$$

The Load is:

$$\lambda = \frac{\Delta M}{\Delta \Phi} = \oint \frac{\rho}{\omega} dl$$

$$\frac{d\lambda}{dt} = 0$$

$$\vec{\nabla}\lambda \cdot \vec{\omega} = 0$$

Stationary flows Lagrangian density

$$\hat{\mathcal{L}} = \rho\left[\frac{1}{2}(\hat{\lambda}\cdot\vec{\nabla}\nu)^2 - \frac{1}{2}(\vec{\nabla}\nu)^2 + B(\lambda) - \varepsilon(\rho)\right]$$

B(λ) is Bernoulli's constant

Stationary Equations

The velocity:

$$\vec{v} = \vec{\nabla}\nu - \hat{\lambda}(\hat{\lambda} \cdot \vec{\nabla}\nu) = \vec{\nabla}^{*}\nu$$

Bernoulli's equation:

$$\frac{1}{2}\vec{v}^{2} + w = B(\lambda)$$

Continuity Equation:

$$\vec{\nabla} \cdot (\rho \vec{v}) = 0$$

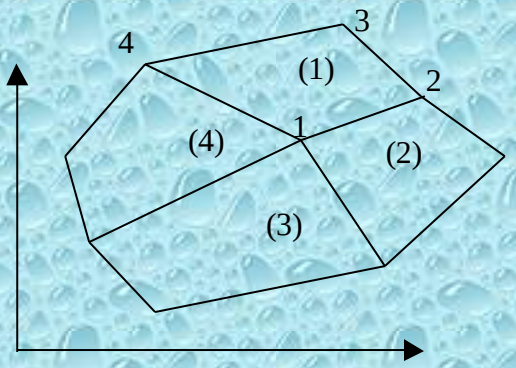
And finally:

$$\rho \left[\frac{dB}{d\lambda} - \vec{v} \cdot \vec{\nabla}^* \left(\frac{\hat{\lambda} \cdot \vec{\nabla} \nu_0}{|\vec{\nabla} \lambda|} \right) \right] = 0$$

Finite Element Approach I

- The action,

$$A \simeq \int dt \sum_{el} \Delta V_{el} \hat{\mathcal{L}}_{el}$$



where

$$\hat{\mathcal{L}}_{el} = \rho_{el} \left[\frac{1}{2} \theta_{el}^2 (\vec{\nabla}\lambda)_{el}^2 - \frac{1}{2} (\vec{\nabla}\nu)_{el}^2 - \left(\frac{\partial \nu}{\partial t}\right)_{el} - \varepsilon(\rho_{el}) \right]$$

Finite Element Approach II

$$\nu \simeq \sum_{i \in el} \nu_{eli} \psi_{eli}, \qquad \lambda \simeq \sum_{i \in el} \lambda_{eli} \psi_{eli}$$

$$\vec{\nabla}\nu \simeq \sum_{i \in el} \nu_{eli} \vec{\nabla}\psi_{eli}, \qquad \vec{\nabla}\lambda \simeq \sum_{i \in el} \lambda_{eli} \vec{\nabla}\psi_{eli}$$

ψ_{eli}

Are interpolation functions.

Finite Element Approach III

$$\hat{\mathcal{L}}_{el} = \rho_{el}\left[\frac{1}{2}\theta_{el}^2(\vec{\nabla}\lambda)_{el}^2 - \frac{1}{2}(\vec{\nabla}\nu)_{el}^2 - (\frac{\partial\nu}{\partial t})_{el} - \varepsilon(\rho_{el})\right]$$

$$(\vec{\nabla}\nu)_{el}^2 \simeq \sum_{i,j\in el} \nu_{eli}\nu_{elj}\vec{\nabla}\psi_{eli}\cdot\vec{\nabla}\psi_{elj} \equiv \sum_{i,j\in el} \nu_{eli}\nu_{elj}\, k_{ij}^{el}$$

$$(\frac{\partial\nu}{\partial t})_{el} \simeq \sum_{i\in el} \frac{d\nu_{eli}}{dt}\psi_{eli}(\bar{\vec{r}}_{el})$$

Finite Element Approach IV

$$\hat{\mathcal{L}}_{el} = \rho_{el}\left[\frac{1}{2}\theta_{el}^2(\vec{\nabla}\lambda)_{el}^2 - \frac{1}{2}(\vec{\nabla}\nu)_{el}^2 - (\frac{\partial \nu}{\partial t})_{el} - \varepsilon(\rho_{el})\right]$$

$$\theta_{el} = -\left(\frac{(\frac{\partial \lambda}{\partial t})_{el} + \vec{\nabla}\lambda_{el}\cdot\vec{\nabla}\nu_{el}}{(\vec{\nabla}\lambda)_{el}^2}\right)$$

$$(\frac{\partial \lambda}{\partial t})_{el} \simeq \sum_{i\in el}\frac{d\lambda_{eli}}{dt}\psi_{eli}(\vec{\bar{r}}_{el})$$

$$(\vec{\nabla}\lambda)_{el}^2 \simeq \sum_{i,j\in el}\lambda_{eli}\lambda_{elj}\,k_{ij}^{el}$$

Conclusions

1. New variational variables and a new variational principle is introduced for Eulerian fluid dynamics, using only three functions.
2. The numerical extremization of the action using the finite element approach will yield an approximate solution to the Euler equations.
3. The solution is give in terms of 2N+EL ordinary differential equations (N=number of nodes, EL=number of elements). In the stationary case those equations become algebraic equations.
4. Moreover, one can skip the algebraic equation stage in the stationary case and thus obtain a solution by using direct numerical extremization of the Lagrangian.