MASS WITHOUT MASS

Solution of the Yang-Mills mass gap problem

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MASS WITHOUT MASS

The concept, realized in modern physics, that objects with non-zero mass can emerge from building blocks of zero mass.

It's a feat worthy of the Cheshire cat.

Frank Wilzcek

Atlanta APS meeting 2012

QUANTUM YANG-MILLS THEORY

(The 7th Millennium problem of Clay Mathematics Institute, 2000)

Find a mathematically complete proof that for any compact semi-simple global gauge group $\mathbf{G}$, a nontrivial quantum Yang-Mills theory exists on the four-dimensional Minkowski space-time and the spectrum of its quantum Hamiltonian has a positive gap.

E.g. $\mathbf{G} = \mathbf{SU}(3)$, $\mathbf{SU}(2)$, $\mathbf{SU}(2)_L \times \mathbf{SU}(2)_R$ for strong and weak (with no supercharge) forces, and for neutrino oscillations.

TROUBLES WITH THE STANDARD MODEL OF ELEMENTARY PARTICLES

9 positive masses of fermionic quarks and electrons,
3 positive masses of weak vector bosons.
Neutrinos are massless.
Yet 2015 Nobel prize in physics is for "The Discovery of Neutrino Oscillations, which shows that Neutrinos have positive Masses".

Too many phenomenological artifacts for a fundamental theory!

CONSTIUENT QUARK MASSES

18 years of Budapest-Marseille-Wuppertal Collaboration produced "Ab-initio Determination of Light Hadron Masses" (Dürr et al, arXiv:0906.3599 [hep-lat]).

The masses of nucleons have been computed correctly via non-perturbative lattice approximations in the framework of the QCD. Only the lightest quarks have been used but 99% of the all visible matter evaluated theoretically!

The bare masses of the lightest quarks are tiny. Only their subnucleonic interaction with the gauge field of gluons have the leading roles.

In 1999 Wilczek has simplified QCD into QCD Lite keeping only lightest quarks stripped of their bare masses. This is not essential for the lattice computations.

QUANTIZATION ORIGIN OF MASS

For pure Yang-Mills theory this is a solution of the Yang-Mills problem. It is published in A. Dynin "Quantum Yang-Mills theory revisited", Russian Journal of Mathematical Physics, 24, (1)(2017), 26-43 where the semi-simplicity of the gauge group is crucial.

The quantum interaction between Majorana fermion fields and Yang-Mills boson fields is the subject of the present lecture.

YANG-MILLS-MAJORANA MASSLESS HAMILTIONIAN

YANG-MILLS FIELDS

A *local gauge group* $\widetilde{\mathbf{G}}$ is the group of $\mathbf{G}$-valued functions $\tilde{g}(x)$ on Minkowki space with coordinates $x = (t, x_1, x_2, x_3) = (t, \vec{x})$.

Yang-Mills fields are relativistic vector fields $A(x) = (A_0, A_1, A_2, A_3)$ with values in the Lie algebra of $\widetilde{\mathbf{G}}$ that satisfy Yang-Mills nonlinear 2nd order partial differential equations.

SYMMETRIES

Yang-Mills equations are invariant under the infinite-dimensional Lie group of local gauge transformations

$$A_\mu \mapsto \tilde{g} A_\mu \tilde{g}^{-1} - (\partial_\mu \tilde{g})\tilde{g}^{-1},$$

and transformations induced by the 10-dimensional group of Lorentz isomeries of the Minkowski space-time along with the 1-dimensional group of scalings

$$A(x) \mapsto sA(sx), \quad 0 < s < \infty.$$

All symmetries are fundamental.

MASSLESS MAJORANA FERMIONS

Solutions Ψ of the Dirac equation which are invariant under under Lorentz invariant complex conjugation $\Psi^c = \Psi$. They form a real vector space.

By the CPT-theorem Majorana fermions are PT are invariant under the induced group of proper Lorentz transformations, the analytic continuation of the compact semi-simple Lie group $\mathbf{SU}(2)_L \times \mathbf{SU}(2)_R$.

YANG-MILLS-MAJORANA FIELDS

Charge-invariant solutions Ψ of the gauged Dirac equations with partial derivatives converted into covariant partial derivatives

$$\partial_{\mu,A}\Psi \;:=\; \partial_\mu\Psi \;+\; [\rho(A),\Psi],$$

where ρ is an orthogonal representation of the gauge group $\mathbf{G}$ into n-multiplets of Majorana fields.

The YMM equation for (A,Ψ) is the system of combined YM-equations for A and gauged Dirac equations for Ψ.

HAMILTONIAN GAUGE $A_0(t,\vec{x}) = 0$

Violates both the local gauge and relativistic invariance of Yang-Mills equations.

Then in the 1st order formalism the partial differential YMM equations split into semi-linear system of Hamiltonan evolution and time-independent nonlinear constraints for

$$A(t,\vec{x}),\ \ E(t,\vec{x}) = \mathrm{curl}\, A + [A \overset{\times}{,} A],\ \ \Psi(t,\vec{x}),$$

where $[\ \overset{\times}{,}\ ]$ is the cross product of vector fields with values in $\widetilde{\mathbf{G}}$.

CAUCHY PROBLEM FOR EVOLUTION

Solutions of the semi-linear partial differential system of Hamiltonian evolution have a finite speed propagation of the Cauchy vector data

$$a = A(0, \vec{x}),\ e = E(0, \vec{x}),\ \psi = \Psi(0, \vec{x}).$$

Thus solutions of the Cauchy problem (with no conditions at the space infinity) are defined by the evolution of initial data with compact supports in $\mathbf{R}^3$.

The evolution of massless Majorana fields is uniquely defined by the Cauchy data in $\vec{x} \cdot \vec{x} < r^2,\ 0 < r < \infty$, and exists at any time.

CONSTRAINT MANIFOLD

In the Hamiltonian gauge the time-independent constraints form a smooth submanifold $\mathcal{C}$ of the vector space of smooth Cauchy data for massless Majorana fields. It carries a global smooth chart $\mathcal{A}_{\mathrm{trs}} \times \mathcal{E}_{\mathrm{trs}} \times \mathcal{P}$ considered as the real diagonal $\mathcal{D}$ in the direct product of the vector spaces of smooth transversal complex vector fields $z := 2^{-1/2}(a_{\mathrm{trs}} + i e_{\mathrm{trs}})$ along with spinor fields ψ and their anti-dual spaces:

$$\mathcal{D}(\mathcal{U} \times \mathcal{V}) := \{(u, v), v = u^*\}.$$

YMM HAMILTONIAN ON $\mathcal{D}$

$$H = H_{g,r}(\psi, z) = (gr)^{-1} H_{1,1} \; =$$

$$\int_{\vec{x}\cdot\vec{x}<r^2} d^3\vec{x}\, [i\psi^\dagger \partial_{k,(z,\bar{z})} \gamma_k \psi - g^{-2}\psi^\dagger\psi],$$

where $0 < r < \infty$, γ_k are Dirac matrices, and g is a dimensionless coupling constant.

The Hamiltonian scalar functional is non-negative on Majorana fermions.

QUANTIZATION OF THE YMM HAMILTONIAN

SUPER HILBERT SPACE $\mathcal{D}^0 = \mathcal{D}^0_{\mathsf{b}} \times \mathcal{D}^0_{\mathsf{f}}$

Bosonic $\mathcal{D}^0_{\mathsf{b}}$ and fermionic $\mathcal{D}^0_{\mathsf{f}}$ are complex Hilbert spaces with conjugation

$$z = (z_{\mathsf{b}}, z_{\mathsf{f}}),\ \overline{z} = (z^*_{\mathsf{b}}, z^*_{\mathsf{f}})$$

Non-degenerate Hermitian super product

$$\overline{z}z \ := \ z^*_{\mathsf{b}} z_{\mathsf{b}} - i\, z^*_{\mathsf{f}} z_{\mathsf{f}} \geq 0$$

(with the bracketless contraction notation) identifies $\mathcal{D}^0$ with its antidual space $\mathcal{D}^{0\dagger}$.

T-RIGGING OF $\mathcal{D}^0$

Let $T = T_b \times T_f$ be a positive selfadjoint operator on $\mathcal{D}^0$ such that T^{-1} has finite trace.

Let $\mathcal{D}^{[k]},\ k = \pm 1, \pm 2, \ldots,$ be the Hilbert spaces with the the Hermitian forms $\overline{z}(1+T)^k z_2$.

The continuous embeddings

$$\mathcal{D} := \cap_{k>0}\mathcal{D}^{[k]} \ \subset \ \mathcal{D}^0 \ \subset \ \mathcal{D}^\dagger := \cup_{k<0}\mathcal{D}^{[k]}$$

form a T-triple where $\mathcal{D}$ is a nuclear Frechet super space.

HILBERT FOCK SPACE $\mathcal{F}(\mathcal{D}^0)$

Let $\overline{z}_i = (\overline{z}_{\mathsf{b}i}, \overline{z}_{\mathsf{f}_j}) \in \mathcal{D}$ form an orthonormal basis for $\mathcal{D}^0$. By acclamation, finite srrictly ordered super products of continuous linear functionals $\overline{z}_{\mathsf{b}i}$ and $\overline{z}_{\mathsf{f}_j}$ on $\mathcal{D}^0$ form an orthonormal basis for the Hilbert Fock space $\mathcal{F}(\mathcal{D}^0)$.

Naturally $\mathcal{F}(\mathcal{D}^0)$ has the continuous Hermitian scalar product and the conjugation defined on the basic products via the reversed factor-wise conjugation.

HOLOMORPHIC RIGGING OF $\mathcal{F}(\mathcal{D}^0)$

$$\mathsf{Exp}(\mathcal{D}^\dagger) \subset \mathcal{F}(\mathcal{D}^0) \subset \mathsf{Ent}(\mathcal{D}),$$

where $\mathsf{Ent}(\mathcal{D})$ is the nuclear space of entire super holomorphic functionals on $\mathcal{D}$, the antidual of the nuclear Frechet space $\mathsf{Exp}(\mathcal{D}^\dagger)$ of functionals from $\mathsf{Ent}(\mathcal{D}^\dagger)$ of the exponential growth type with respect to a continuous seminorm on $\mathcal{D}^\dagger$.

CREATORS AND ANNIHILATORS

The coherent states

$$e^z(\zeta^*) := e^{\zeta^* z}, \quad z, \zeta \in \mathcal{D},$$

form a continual basis in $\mathrm{Exp}(\mathcal{D}^\dagger)$. Thus the component-wise creators and annihilators

$$z^+ := (z\zeta^*)\Psi(\zeta^*), \quad \zeta^{*-} := \partial_{\zeta^*}\Psi(\zeta^*)$$

are well defined adjoint operators in $\mathrm{Exp}(\mathcal{D}^\dagger)$ and their super commutators are equal to the super contractions:

$$[\zeta^{*-}, z^+] = \zeta^* z.$$

COHERENT STATES MATRIX ELEMENTS

The vector space of continuous linear operators $\mathcal{F}(\mathcal{D}^\dagger) \to \mathcal{F}(\mathcal{D})$ is, by the nuclearity, topologically isomorphic to the vector space $\mathcal{F}(\overline{\mathcal{D}} \times \mathcal{D})$ of the coherent states matrix elements

$$\Theta^Q(\overline{z}, \zeta) := \langle e^{\overline{z}} \ |Q| \ e^{\zeta} \rangle, \ (\overline{z}, \zeta) \in \overline{\mathcal{D}} \times \mathcal{D}).$$

Actually if $\Theta^Q(\overline{z}, \zeta)$ is a continuous polynomial then

$$Q : \mathcal{F}(\mathcal{D}^\dagger) \to \mathcal{F}(\mathcal{D}^0).$$

NORMAL, WEYL, ANTINORMAL SYMBOLS OF Q

They are the real-analytic functionals on the real diagonal $\{(\overline{z}, \zeta = z)\}$ of $\overline{\mathcal{D}} \times \mathcal{D}$

$$\begin{aligned}
\theta^Q_\nu(\overline{z}) &:= e^{-\overline{z}z} \Theta^Q(\overline{z}, z), \\
\theta^Q_\omega(\overline{z}) &:= e^{-\partial_{\overline{z}}\partial_z/2}\, \theta^Q_\nu(\overline{z}, z), \\
\theta^Q_\alpha(\overline{z}) &:= e^{-\partial_{\overline{z}}\partial_z/2}\, \theta^Q_\omega(\overline{z}, z),
\end{aligned}$$

where super contractions are applied.

The correspondences $\theta^Q \leftrightarrow Q$ are one-to-one.

SPECTRAL THEORY

If the antinormal symbol θ_α^Q is polynomial and non-negative then $Q \geq 0$ and has a unique self-adjoint Friedrichs extension $\mathrm{Fr}(Q)$ in the Hilbert space $\mathcal{F}^0 = \mathcal{F}^{*0}$.

Furthermore the infimum of the non-negative spectrum is $\leq \theta_\nu^Q(0)$ and $\geq \inf \theta_\alpha^Q$.

QUANTIZED YMM HAMILTONIAN

Let $\mathcal{D}^0$ be the complex Hilbert super space with conjugation, where $z_b = 2^{-1/2}(a + ie)$ are Yang-Mills fields and $z_f = \psi$ are complexified Majorana spinor fields with compact supports on the unit ball $\vec{x} \cdot \vec{x} < r^2$.

Let $T = \Delta$ so that the non-negative inverse $(1 + T)^{-1}$ has the finite trace.

Since YMM Hamiltonian is a continuous polynomial $H_{g,r}$ on $\overline{\mathcal{D}} \times \mathcal{D}$, it is the anti-normal symbol of a unique positive operator $\widehat{H_{g,r}}$.

QUANTUM YANG-MILLS-MAJORANA MASS GAP

THE SPECTRUM OF THE QUANTiZED YMM HAMILTONIAN $\widehat{H_{g,r}}$ HAS A POSITIVE MASS GAP $\geq (2g^2r)^{-1}$.

Thus the mass gap grows to infinity as $r \to 0$ (*asymptotic freedom*?).

For strong quark interactions inside a nucleon $g, r \approx 1$ so that the total mass of the 3 constituent quarks is $> 3/2$ (*quarks confinement*?).

MASS GAP ESTIMATE FOR $\widehat{H_{1,1}}$

The Weyl symbol $\theta_\omega^{\widehat{H_{1,1}}}$ has a lower bound $\overline{z}z - 1/2$, the Weyl symbol of the number operator N, plus $3/4$. The number operator has the simple zero fundamental eigenvalue and the mass gap 1. Then, by a variational spectral principle, $\widehat{H_{1,1}}$ has no eigenvalues between its (simple) fundamental eigenvalue and $1 + 3/4$.
On the other hand, the constant term $3/2$ of the normal symbol $\theta_\nu^{\widehat{H_{1,1}}}$ is the upper bound for the fundamental eigenvalue of $\widehat{H_{1,1}}$.