



Tomorrow's Science Fractional Calculus View of Complexity

Presented at
PHYSICS 2016
Video Presentation
May 25, 2016



ARL

Bruce J. West

ST- Chief Scientist Mathematics
Army Research Office
bruce.j.west.civ@mail.mil

919-549-4257

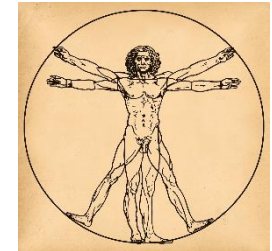
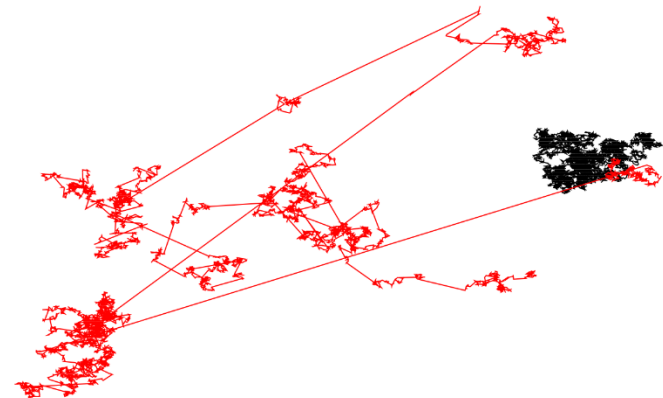
Collaborators: **P. Grigolini**
M. Bologna
M. Turala
A. Svenkeson
M.T. Beig
P. Pramukkul

- Why a fractional calculus?
 - new ways of thinking
 - dynamics and fractals
- Fractional dynamics
 - fractional difference equations
 - simple fractional operators
 - fractional rate equation
- Fractional diffusion and probability
 - turbulent diffusion
 - fractional Bloch equation
 - Lévy foraging
 - phase space fractional equations
- Conclusions

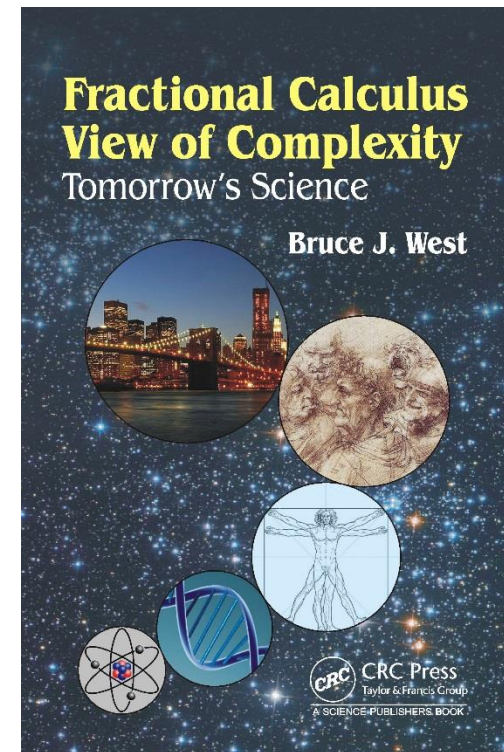
Euclid
Classical geometry



Mandelbrot
Fractal geometry



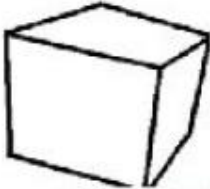
- Fractional thinking is in-between thinking:
 - between integers there are non-integers
 - between integer-order moments there are fractional moments
 - between integer dimensions there are fractal dimensions
 - between integer Fourier series are fractional Fourier transforms
 - between integer-valued operators are fractional-order operators
- This tutorial is on how the fractional calculus provides insight into complex dynamic networks.
- Complexity is emphasized, which highlights the inability of traditional analytic functions to satisfactorily characterized the rich structure of complex dynamic phenomena (networks) in both space and time.



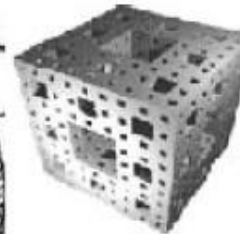
2015

....A NEW WAY OF THINKING....

old

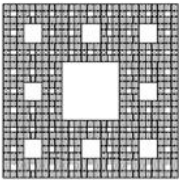


new

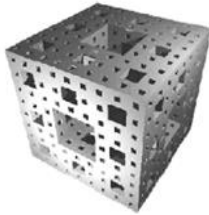


Why is the fractional calculus entailed by complexity?

...from integer to non-integer...

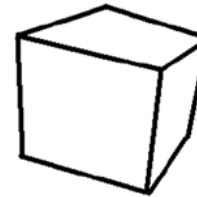


$$D = 1.89$$



$$D = 2.73$$

$$D = 3$$



$$D = 2$$



$$D = 1.26$$

$$D = 1$$

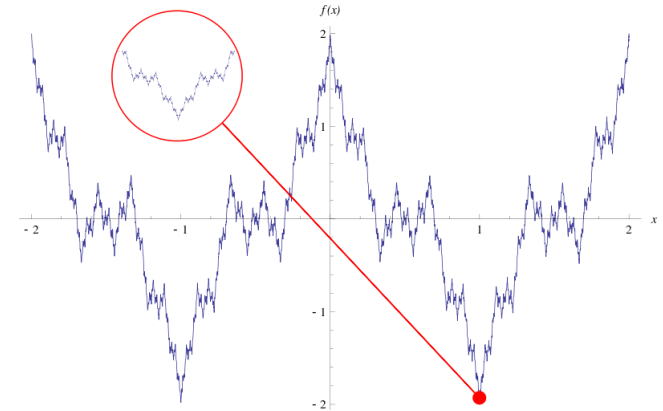


How pervasive are non-integer phenomena?

....no equations of motion....

- Karl Weierstrass (1872): generalized by Mandelbrot (1977)

$$W(t) = \sum_{n=-\infty}^{\infty} \frac{1}{a^n} [1 - \cos(b^n \omega_0 t)]; \quad b > a > 1$$



- Interesting properties

- continuous everywhere

- nowhere differentiable

- self-similar

$$\frac{dW(t)}{dt} = \sum_{n=-\infty}^{\infty} \left(\frac{b}{a}\right)^n \cos(b^n \omega_0 t) = \infty$$

$$W(bt) = aW(t) \Rightarrow W(t) \propto t^\mu; \quad \mu = \frac{\log a}{\log b}$$

- What are the dynamic equations for fractal functions?

....Turbulent diffusion....

- Richardson at the London Expo, released 10,000 balloons with a return address. From the data on where/when the balloons landed he constructed *Richardson Dispersion Law*

$$\frac{d\langle R(t)^2 \rangle}{dt} = \langle R(t)^2 \rangle^{2/3}$$

- The solution yields the lateral growth of smoke plumes

$$\langle R(t)^2 \rangle \propto t^3$$

- Molecular diffusion has a mean-square displacement

$$\langle R(t)^2 \rangle \propto t$$

- Anomalous diffusion was therefore first observed in the study of turbulent fluid flow.
- Perhaps it could be described by a Weierstrass function?



Empirical Power Laws

Discipline	Law's name	Form of law
Anthropology		
1913 [4]	Auerbach	$\text{Pr}(\text{city size rank } r) \propto 1/r$
1998 [65]	War	$\text{Pr}(\text{intensity} > I) \propto 1/I^\alpha$
1978 [86]	1/f Music	$\text{Spectrum}(f) \propto 1/f$
Biology		
1992 [87]	DNA sequence	$\text{Symbol spectrum}(\text{frequency } f) \propto 1/f^\alpha$
2000 [49]	Ecological web	$\text{Pr}(k \text{ species connections}) \propto 1/k^{1.1}$
2001 [35]	Protein	$\text{Pr}(k \text{ connections}) \propto 1/k^{2.4}$
2000 [34]	Metabolism	$\text{Pr}(k \text{ connections}) \propto 1/k^{2.2}$
2001 [40]	Sexual relations	$\text{Pr}(k \text{ relations}) \propto 1/k^\alpha$
Botany		
1883 [64]	da Vinci	Branching; $d_0^\alpha = d_1^\alpha + d_2^\alpha$
1922 [101]	Willis	No. of genera (No. of species N) $\propto 1/N^\alpha$
1927 [51]	Murray	$d_0^{2.5} = d_1^{2.5} + d_2^{2.5}$
Economics		
1897 [56]	Pareto	$\text{Pr}(\text{income } x) \propto 1/x^{1.5}$
1998 [24]	Price variations	$\text{Pr}(\text{stock price variations } x) \propto 1/x^3$
Geophysics		
1894 [55]	Omori	$\text{Pr}(\text{aftershocks in time } t) \propto 1/t$
1933 [67]	Rosen–Rammler	$\text{Pr}(\text{No. of ore fragments} < \text{size } r) \propto r^\alpha$
1938 [44]	Korčák	$\text{Pr}(\text{island area } A > a) \propto 1/a^\alpha$
1945 [31]	Horton	No. of segments at n /No. of segments at $n + 1 = \text{constant}$
1954 [26]	Gutenberg–Richter	$\text{Pr}(\text{earthquake magnitude} < x) \propto 1/x^\alpha$
1957 [27]	Hack	River length $\propto (\text{basin area})^\alpha$
1977 [44]	Richardson	Length of coastline $\propto 1/(\text{ruler size})^\alpha$
2004 [84]	Forest fires	Frequency density (burned area A) $\propto 1/A^{1.38}$
Information theory		
1999 [32]	World Wide Web	$\text{Pr}(k \text{ connections}) \propto 1/k^{1.94}$
1999 [19]	Internet	$\text{Pr}(k \text{ connections}) \propto 1/k^\alpha$

Discipline	Law's name	Form of law
Physics		
1918 [70]	1/f noise	$\text{Spectrum}(f) \propto 1/f$
2002 [25]	Solar flares	$\text{Pr}(\text{time between flares } t) \propto 1/t^{2.14}$
2003 [69]	Temperature anomalies	$\text{Pr}(\text{time between events } t) \propto 1/t^{2.14}$
Physiology		
1959 [61]	Rall	Neurons; $d_0^{1.5} = d_1^{1.5} + d_2^{1.5}$
1963 [76]	Mammalian vascular network	Veins and arteries; $d_0^{2.7} = d_1^{2.7} + d_2^{2.7}$
1963 [90]	Bronchial tree	$d_0^3 = d_1^3 + d_2^3$
1973 [48]	McMahon	Metabolic rate (body mass M) $\propto M^{0.75}$
1976 [103]	Radioactive clearance	$\text{Pr}(\text{isotope expelled in time } t) \propto 1/t^\alpha$
1987 [93]	West–Goldberger	Airway diameter (generation n) $\propto 1/n^{1.25}$
1991 [30]	Mammalian brain	Surface area $\propto \text{volume}^{0.90}$
1992 [77]	Interbreath variability	No. of breaths (interbreath time t) $\propto 1/t^{2.16}$
1993 [58]	Heartbeat variability	Power spectrum (frequency f) $\propto f$
2007 [23]	EEG	$\text{Pr}(\text{time between EEG events}) \propto 1/t^{1.61}$
2007 [13]	Motivation and addiction	$\text{Pr}(k \text{ behavior connections}) \propto 1/k^\alpha$
Psychology		
1957 [75]	Psychophysics	Perceived response (stimulus intensity x) $\propto x^\alpha$
1963 [71]	Trial and error	Reaction time (trial N) $\propto 1/N^{0.91}$
1961 [29]	Decision making	utility (delay time t) $\propto 1/t^\alpha$
1991 [3]	Forgetting	Percentage correct recall (time t) $\propto 1/t^\alpha$
2001 [20]	Cognition	Response spectrum (frequency f) $\propto 1/f^\alpha$
2009 [37]	Neurophysiology	$\text{Pr}(\text{phase-locked interval} < \tau) \propto 1/\tau^\alpha$
Sociology		
1926 [41]	Lotka	$\text{Pr}(\text{No. of papers published rank } r) \propto 1/r^2$
1949 [104]	Zipf	$\text{Pr}(\text{word has rank } r) \propto 1/r$
1963 [16]	Price	$\text{Pr}(\text{citation rank } r) \propto 1/r^3$
1994 [8]	Urban growth	Population density (radius R) $\propto 1/R^\alpha$
1998 [88]	Actors	$\text{Pr}(k \text{ connections}) \propto 1/k^{2.3}$

- Physics: constitutive relationships

- Hooks law in ideal solids

$$F = kx$$

- Ideal Newtonian fluid

$$F = v \frac{\partial u}{\partial y}$$

- Newton's law of motion

$$F = m \frac{d^2 x}{dt^2}$$

- One model for soft matter

$$F = \rho \frac{d^\alpha x}{dt^\alpha}; \quad 0 \leq \alpha \leq 2$$



Integer-order calculus



Fractional-order calculus

- Example of Riemann-Liouville fractional derivative; using properties of Gamma functions.

$$\frac{d^\alpha}{dt^\alpha} [t^\beta] = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha}$$

$$\frac{d^{1/2}}{dt^{1/2}} [t^{-1/2}] = 0$$

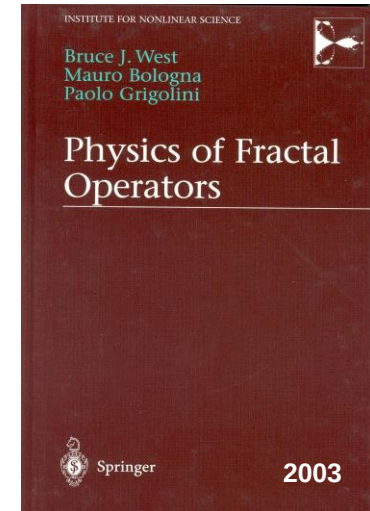
$$\frac{d^{1/2}}{dt^{1/2}} [1] = \frac{1}{\sqrt{\pi t}}$$

$$\frac{d^{1/2}}{dt^{1/2}} [t] = \sqrt{\frac{t}{\pi}}$$



Curious results not consistent with ordinary calculus

Result obtain by Leibniz in response to question by L'Hopital
In 1695.



- One way to capture complex dynamics
- Rate equation:

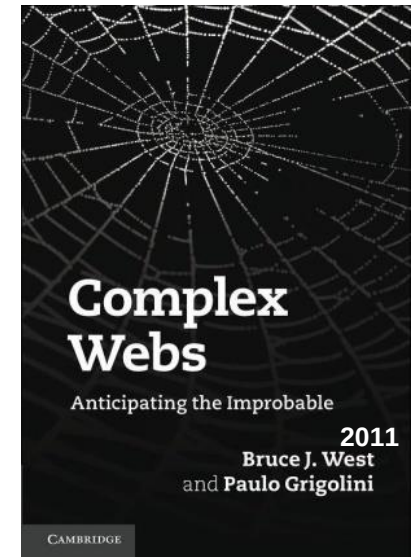
$$\frac{dQ(t)}{dt} = -\lambda Q(t) \Rightarrow Q(t) = Q(0)e^{-\lambda t}$$

- Fractional rate equation (FRE):

$$\frac{d^\alpha Q(t)}{dt^\alpha} = -\lambda Q(t) \Rightarrow Q(t) = ? \text{ for } \alpha \neq \text{integer}$$

- Caputo fractional derivative: defined in terms of Laplace transform

$$LT \left\{ \frac{d^\alpha Q(t)}{dt^\alpha}; u \right\} = u^\alpha \hat{Q}(u) - u^{\alpha-1} Q(0)$$



- Laplace solution to fractional rate equation:

$$u^\alpha \hat{Q}(u) - u^{\alpha-1} Q(0) = -\lambda \hat{Q}(u) \Rightarrow \hat{Q}(u) = \frac{u^{\alpha-1}}{u^\alpha + \lambda} Q(0)$$

- Inverse Laplace transform:

$$LT^{-1}\{\hat{Q}(u); t\} = Q(t) = Q(0)E_\alpha(-\lambda t^\alpha)$$

- Solution first obtain by Mittag-Leffler in 1903:

$$E_\alpha(-\lambda t^\alpha) = \sum_{k=0}^{\infty} \frac{(-\lambda t^\alpha)^k}{\Gamma(k\alpha + 1)}$$

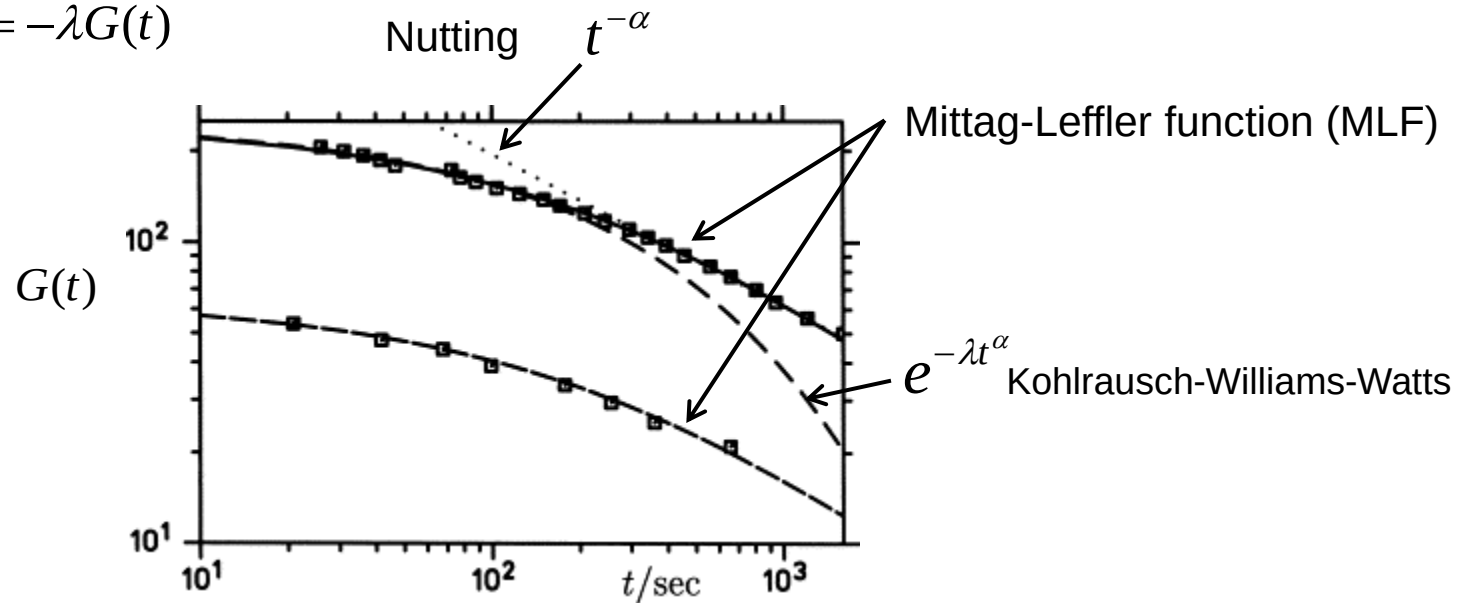
$\swarrow$
 $\searrow$

$\exp[-\lambda t^\alpha]$ as $t \rightarrow 0$

$\frac{1}{\lambda t^\alpha}$ as $t \rightarrow \infty$

- Viscoelastic material experiments: generalized stress-strain relations
 - Relaxation function $G(t)$:

$$\frac{d^\alpha G(t)}{dt^\alpha} = -\lambda G(t)$$



- Stress relaxation: fractional MLF smoothly joins two empirical laws
Glöckle & Nonnenmacher (*J. Stat. Phys.* **71**,1993; *Biophys. J.* **68**, 1995)

- A second reason to learn the fractional calculus
- Consider the Caputo fractional derivative of the Generalized Weierstrass Function whose Laplace transform is

$$LT\left\{\frac{d^\alpha W(t)}{dt^\alpha}; u\right\} = u^\alpha \hat{W}(u) = \sum_{n=-\infty}^{\infty} \frac{(b^n \omega_0)^2}{a^n} \frac{u^{\alpha-1}}{u^2 + (b^n \omega_0)^2}$$

- No analytic inverse but the inverse Laplace transform does scale

$$W^\alpha(t) = LT^{-1}\left\{u^\alpha \hat{W}(u); t\right\} \qquad W^\alpha(bt) = \frac{a}{b^\alpha} W^\alpha(t) \Rightarrow D = \mu - \alpha$$

★ Fractional derivative α of fractal function of dimension μ is another fractal function with fractal dimension $\mu - \alpha$; it does not diverge. Fractional calculus yields the appropriate dynamics for fractal processes.

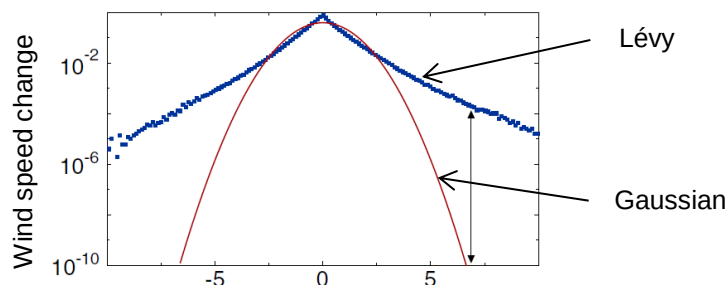
- Fractional Probability Density

$$\partial_t^\beta P(x,t) = K_\alpha \partial_{|x|}^\alpha P(x,t) \xrightarrow{\beta=1, FT} \partial_t \hat{P}(k,t) = -K_\alpha |k|^\alpha \hat{P}(k,t)$$

- α -stable Lévy distribution

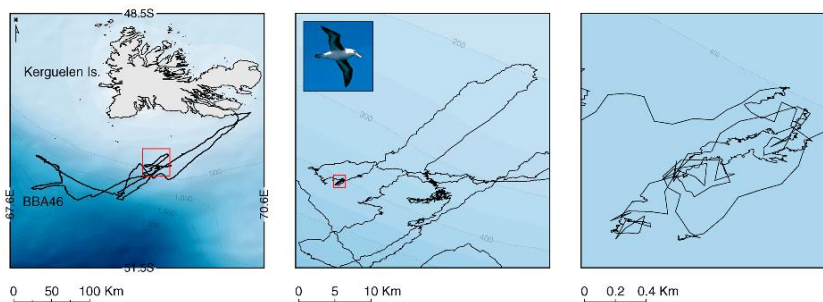
$$P_\alpha(x,t) = FT^{-1} \{ \hat{P}(k,t); x \} = FT^{-1} \left\{ \exp \left[-K_\alpha |k|^\alpha t \right] x \right\}$$

- Fractional Turbulence



Boettcher et al., *Boundary-Layer Meteorology* **108** (2003)

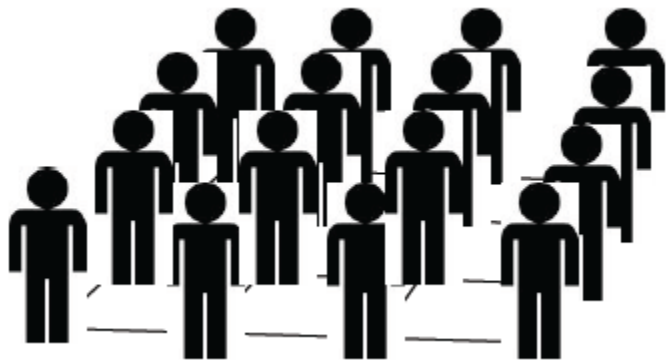
- Lévy Foraging (human mobility)



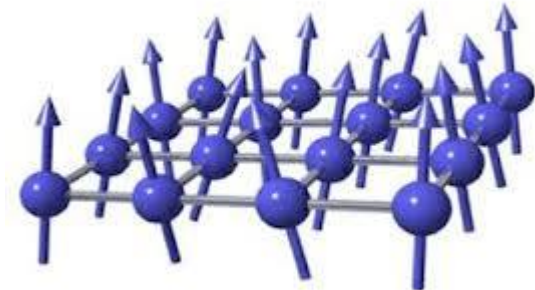
Humphries et al., *Nature* **465** (2012).

- Another approach to the fractional calculus

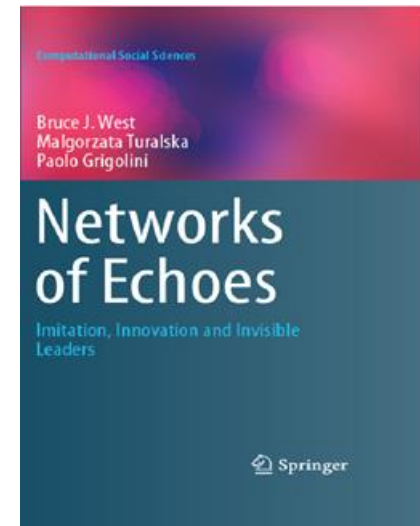
Human network



Network model



- Two-state master equation decision making model (DMM)
- DMM is member of Ising universality class
 - phase transitions to consensus
 - scaling behavior
 - temporal complexity
- How does the network dynamics influence individual dynamics?



- **Subordination** models numerical integration of individual opinion $s(n)$ in discrete operational time n :

$$s(n+1) = (1 - \Delta\tau g)s(n) \Rightarrow s(n) = (1 - \Delta\tau g)^n s_0$$

- This is the time experienced by the individual and for $g\Delta\tau \ll 1$ is a Poisson process
- The influence of network dynamics on individual in chronological time t is (*The Organization Man*)

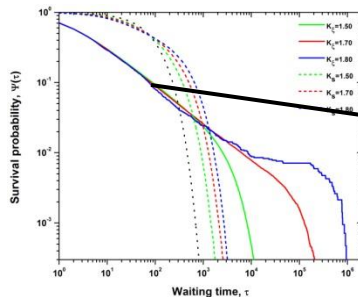
$$\langle s(t) \rangle = \sum_{n=0}^{\infty} \int_0^t \psi_n(t') \Psi(t-t') s(n) dt'$$

Pramukul, Svenkeson, Grigolini, Bologna & West,
Advances in Mathematical Physics 2013, Article ID 498789
(2013).

Probability density of
last of n events occurs
in time $(0, t)$

Probability no event
occurs in $(t-t')$

- The waiting-time distribution and survival probability are taken from numerics.



$$\Psi(t) = \left(\frac{T}{T+t} \right)^{\mu-1} \Rightarrow \psi(t) = (\mu-1) \frac{T^{\mu-1}}{(T+t)^{\mu}}$$

Turalska & West, *Chaos, Solitons & Fractals* **55**, 109 (2013)

- Solve the subordination equation using Laplace transforms to obtain fractional differential equation for average individual opinion:

$$\frac{d^{\alpha}}{dt^{\alpha}} \langle s(t) \rangle = -\lambda \langle s(t) \rangle ; \alpha = \mu - 1$$

- This is the **predicted average dynamics** of the single element within the social network.

- Average opinion $\langle s(j,t) \rangle = p_+(j,t) - p_-(j,t)$

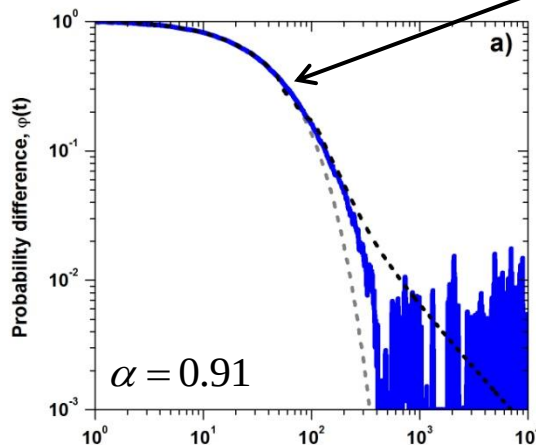
$$\frac{d^\alpha}{dt^\alpha} \langle s(t) \rangle = -\lambda \langle s(t) \rangle \quad \text{Fractional Differential Equation (FDE)}$$

- Solution to FDE is the Mittag-Leffler function

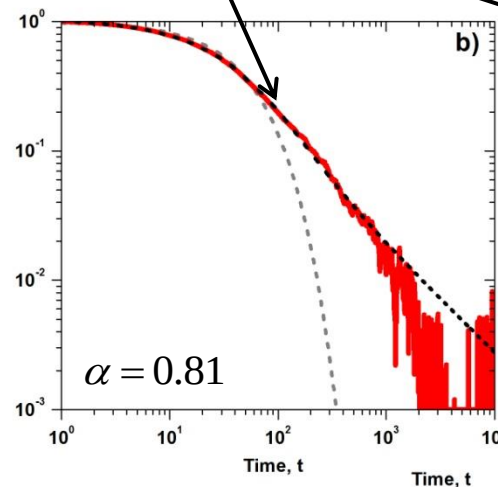
$$\langle s(t) \rangle = E_\alpha(-\lambda t^\alpha) = \sum_{n=1}^{\infty} \frac{(-\lambda t^\alpha)^n}{\Gamma(n\alpha + 1)}$$

Network Effect

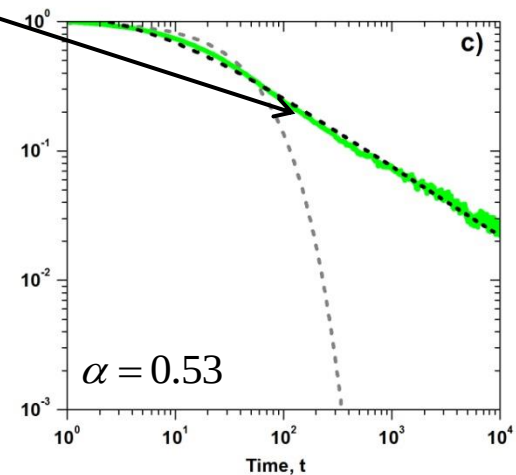
$$r^2 \geq 0.99$$



$$K \leq K_C$$



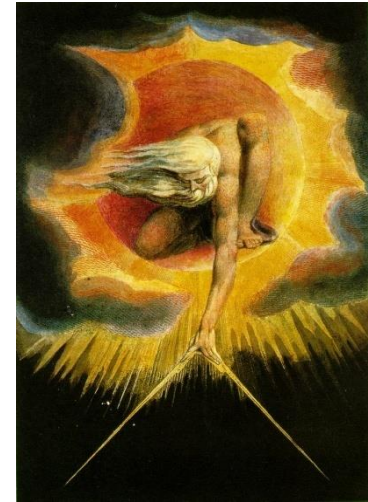
$$K = K_C$$



$$K \geq K_C$$

Conclusions

- The fractional calculus provides a new perspective on complexity.
- It has been used to describe the dynamics of turbulent and anomalous diffusion, optimal foraging, viscoelastic relaxation, and on and on
- The fractional calculus provides a framework for the dynamics of scale-free complex networks.
- The influence of a network on an individual is described by a **stochastic fractional differential equation**.
- Network dynamics transforms a Poisson-type individual into a Mittag-Leffler-type person (the **Network Effect**).



2011

2014

2016

